# VII.—*On a certain Multiple Definite Integral.* *By* GEORGE BOOLE, Esq.

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1. IT has for some time been known that the evaluation of multiple definite integrals can in many cases be effected by the employment of discontinuous functions. M. Lejeune Dirichlet has, in this way, obtained the value of the multiple integral,

$$\iint \cdot \cdot \frac{dx_1 dx_2 \ldots x^{b-1} y^{m-1}}{[(a_1 - x_1)^2 + (a_2 - x_2)^2 \ldots]^i}$$

the limits being given by the inequality,

$$\frac{x_1^{\,2}}{h_1^{\,2}} + \frac{x_2^{\,2}}{h_2^{\,2}}, \ldots \overline{\angle}\ 1.$$

He founds his analysis on the properties of the discontinuous function,

$$\int_0^\infty \frac{d\phi \sin \phi \cos r\phi}{\phi},$$

which is equal to $\frac{\pi}{2}$ when $r$ lies within the limits $-1$ and $1$, and vanishes when $r$ transcends those limits. As a process of integration, M. Dirichlet's is, perhaps, the most remarkable that has ever been published, but it does not appear that any account of it has been published in the English language. For his own knowledge of it the author of this paper is indebted to an abstract of M. Dirichlet's memoir, with which he was favoured by Mr. Cayley, whose investigations on this subject* are scarcely less important, though pursued by a different method.

* Cambridge Mathematical Journal, Nos. XIV. XV.

The idea of the employment of discontinuous functions seems also to have independently occurred to Mr. Ellis,* who makes use of Fourier's theorem, and is thus led to some elegant results, which M. Dirichlet's process would fail to discover. We believe it may ultimately be found that all definite multiple integrals of which the finite evaluation is possible, may be resolved into distinct classes, each dependent on some primary discontinuous function. In the example subjoined we shall make use of the theorem,

$$\frac{f(x)}{t^n}=\frac{1}{\pi\Gamma(n)}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}da\,dv\,dw\cos\left\{(a-x)v-tw+n\frac{\pi}{2}\right\}w^{n-1}f(a), \qquad (1)$$

in which $f(x)$ may be discontinuous; and the first sign of integration in the second member is to be used in the same way as the corresponding one in Fourier's theorem. The two remaining signs are used in the "limiting" sense, according to which $\int_0^\infty dv\phi(v)$, $\int_0^\infty dw\phi(w)$, are considered as the limits of $\int_0^\infty dv\epsilon^{-kv}\phi(v)$, $\int_0^\infty dw\epsilon^{-kw}\phi(w)$; $k$ being a positive quantity approximating to 0.

2. It is required to evaluate the definite multiple integral

$$\text{v}=\iint..\,dx_1dx_2..\frac{f\left(\frac{x_1^2}{h_1^2}+\frac{x_2^2}{h_2^2}..\right)}{[(a_1-x_1)^2+(a_2-x_2)^2..]^i} \qquad (2)$$

the number of the variables being $n$, and the integrations limited by the condition

$$\frac{x_1^2}{h_1^2}+\frac{x_2^2}{h_2^2}\dots \overline{\angle}\ 1. \qquad (3)$$

We may remark that the value of this integral cannot be deduced from the formulæ of M. Dirichlet and others, although in some cases such generalizations may be effected.

Since $\frac{x_1^2}{h_1^2}+\frac{x_1^2}{h_2^2}..$ is a discontinuous function lying between 0 and 1, we have by (1),

$$\frac{f\left(\frac{x_1^2}{h_1^2}+\frac{x_2^2}{h_2^2}..\right)}{\{(a_1-x_1)^2+(a_2-x_2)^2..\}^i}=\frac{1}{\pi\Gamma(i)}\int_0^1\int_0^{\infty}\int_0^{\infty}da\,dv\,dw\cos(\text{T})w^{i-1}f(a), \qquad (4)$$

* Cambridge Mathematical Journal, Nos. XIX. XX. XXI.

in which, for abbreviation,

$$\text{T} = \left(a - \frac{x_1^2}{h_1^2} - \frac{x_2^2}{h_2^2} \ldots\right) v - \{(a_1 - x_1)^2 + (a_2 - x_2)^2 \ldots\} w + i\frac{\pi}{2}, \tag{5}$$

and since the above expression vanishes whenever $\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} \ldots$ transcends the limits assigned in (3), it is evident, that on substitution in (2), we may extend the limits of the integrations relative to $x_1, x_2 \ldots$ from $-\infty$ to $\infty$, and suppose them to be taken in the limiting sense above explained. Hence we have

$$\text{V} = \frac{1}{\pi\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty dadvdw\, w^{i-1} f(a) \int_{-\infty}^{\infty} \ldots dx_1 dx_2 \ldots \cos(\text{T}). \tag{6}$$

We shall first consider the expression

$$\int_{-\infty}^{\infty} \ldots dx_1 \ldots \cos(\text{T}).$$

Now, arranging T with reference to the suffixes,

$$\int_{-\infty}^{\infty} \ldots dx_1 \ldots \cos(\text{T}) = \int_{-\infty}^{\infty} \ldots dx_1 \ldots \cos\left(av - \frac{x_1^2}{h_1^2} v - (a_1 - x_1)^2 w - \ldots + i\frac{\pi}{2}\right).$$

But

$$\frac{vx_1^2}{h_1^2} + w(a_1 - x_1)^2 = \frac{v + wh_1^2}{h_1^2}\left(x_1 - \frac{awh_1^2}{v + wh_1^2}\right)^2 + \frac{a_1^2 vw}{v + h_1^2 w}.$$

Let $x_1 - \frac{a_1 wh_1^2}{v + wh_1^2} = u_1$, the limits of $u_1$, are $-\infty$ and $\infty$, and substituting in a similar manner for the other variables, we have

$$\int_{-\infty}^{\infty} \ldots dx_1 \ldots \cos(\text{T}) = \int_{-\infty}^{\infty} \ldots du_1 \ldots \cos\left(av - \left(\frac{a_1^2}{v + h_1^2 w} + \ldots\right) vw + i\frac{\pi}{2} - \frac{v + h_1^2 w}{h_1^2} u_1^2 \ldots\right). \tag{7}$$

Now by successive applications of the theorem,

$$\int_{-\infty}^{\infty} du_1 \cos(a \pm ru_1^2) = \frac{\pi^{\frac{1}{2}}}{r^{\frac{1}{2}}} \cos\left(a \pm \frac{\pi}{4}\right),$$

we find

$$\int_{-\infty}^{\infty} \ldots du_1 \ldots du_n \cos(a - r_1 u_1^2 \ldots - r_n u_n^2) = \frac{\pi^{\frac{n}{2}}}{(r_1 r_2 \ldots r_n)^{\frac{1}{2}}} \cos\left(a - n\frac{\pi}{4}\right). \tag{8}$$

Integrating by this formula the second member of (7),

$$\int_{-\infty}^{\infty} \ldots dx_1 \ldots \cos(\mathrm{T}) = \frac{\pi^{\frac{n}{2}}}{\left\{\frac{v+h_1^2 w}{h_1^2} \frac{v+h_2^2 w}{h_2^2} \ldots\right\}^{\frac{1}{2}}} \cos\left(av - \left(\frac{a_1^2}{v+h_1^2 w} - \ldots\right)vw + \frac{2i-n}{4}\pi\right)$$

and substituting in (6),

$$\mathrm{V} = \frac{\pi^{\frac{n}{2}-1}}{\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty dadvdw \frac{w^{i-1}}{\left\{\frac{v+h_1^2 w}{h_1^2} \ldots\right\}^{\frac{1}{2}}} \cos\left(av - \left(\frac{a_1^2}{v+h_1^2 w} + \ldots\right)vw + \frac{2i-n}{4}\pi\right) f(a) =$$

$$\frac{h_1 h_2 \ldots \pi^{\frac{n}{2}-1}}{\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty dadvdw \frac{w^{i-1}}{[(v+h_1^2 w) \ldots]^{\frac{1}{2}}} \cos\left(av - \left(\frac{a_1^2}{v+h_1^2 w} + \ldots\right)vw + \frac{2i-n}{4}\pi\right) f(a). \quad (9)$$

Let $w = \frac{v}{s}$, then $dw = -\frac{v}{s^2} ds$, and the limits of $s$ are 0 and $\infty$, whence,

$$\mathrm{V} = \frac{h_1 h_2 \ldots \pi^{\frac{n}{2}-1}}{\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty dadvds \frac{v^{i-\frac{n}{2}}}{s^{i-\frac{n}{2}+1} [(s+h_1^2)(s+h_2^2) \ldots]^{\frac{1}{2}}} \cos\left(av - \left(\frac{a_1^2}{s+h_1^2} \ldots\right)v + \frac{2i-n}{4}\pi\right) f(a)$$

$$= \frac{h_1 h_2 \ldots \pi^{\frac{n}{2}-1}}{\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty dadvds \frac{v^{i-\frac{n}{2}}}{s^{i-\frac{n}{2}+1} \phi^{\frac{1}{2}}} \cos\left((a-\sigma)\, v + \frac{2i-n}{4}\pi\right) f(a). \quad (10)$$

if, for simplicity, we write

$$\left.\begin{aligned} \phi &= (s+h_1^2)\,(s+h_2^2) \ldots (s+h_n^2) \\ \sigma &= \frac{a_1^2}{s+h_1^2} + \frac{a_2^2}{s+h_2^2} \ldots + \frac{a_n^2}{s+h_n^2} \end{aligned}\right\} \quad (11)$$

Now $\cos\left((a-\sigma)\, v + \frac{2i-n}{4}\pi\right) v^{i-\frac{n}{2}} = (-)^{i-\frac{n}{2}} \left(\frac{d}{d\sigma}\right)^{i-\frac{n}{2}} \cos(a-\sigma)v.$

Substituting this value in (10), we may express the result in the following form,

$$\mathrm{V} = \frac{h_1 h_2 \ldots \pi^{\frac{n}{2}}}{\Gamma(i)} \int_0^\infty ds \frac{(-)^{i-\frac{n}{2}} \left(\frac{d}{d\sigma}\right)^{i-\frac{n}{2}} \mathrm{Q}}{s^{i-\frac{n}{2}+1} \phi^{\frac{1}{2}}},$$

wherein $\mathrm{Q} = \frac{1}{\pi} \int_0^1 \int_0^\infty dadv \cos(a-\sigma)v f(a)$. But by Fourier's theorem this value of Q is equivalent to $f(\sigma)$, when $\sigma$ lies between 0 and 1, and is equal to 0 when $\sigma$ transcends those limits. Hence, substituting for Q, we finally get

$$\mathrm{v} = (-)^{i-\frac{n}{2}} \frac{h_1 h_2 \ldots h_n \pi^{\frac{n}{2}}}{\Gamma(i)} \int_0^\infty \frac{ds \left(\frac{d}{d\sigma}\right)^{i-\frac{n}{2}} f(\sigma)}{s^{i-\frac{n}{2}+1} \phi^{\frac{1}{2}}}, \tag{12}$$

$\phi$ and $\sigma$ being the symmetrical functions of $s$ given in (11), and $f(\sigma)$ being replaced by 0 when $\sigma > 1$.

The reader who is acquainted with the researches of M. Dirichlet and Mr. Ellis, will, in various parts of the preceding investigation, be reminded of them, and I am anxious to express in the fullest manner my obligations to them. But I think it will be evident, that while the methods of those writers apply to classes of questions which are mutually distinct, neither of them is, in its actual state, applicable to the question of this paper.

## PARTICULAR DEDUCTIONS.

1st. Let $n = 3$, $i = \frac{1}{2}$, and let us write $xyz$ for $x_1 x_2 x_3$, and $abc$ for $a_1 a_2 a_3$, we have

$$\mathrm{v} = \iiint \frac{dx\,dy\,dz\, f\left(\frac{x^2}{h_1^2} + \frac{y^2}{h_2^2} + \frac{z^2}{h_3^2}\right)}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{\frac{1}{2}}} \tag{13}$$

subject to the inequality,

$$\frac{x^2}{h_1^2} + \frac{y^2}{h_2^2} + \frac{z^2}{h_3^2} \overline{\gtrless} 1; \tag{14}$$

and its value given by (12) is

$$\mathrm{v} = -\, h_1 h_2 h_3 \pi \int_0^\infty \frac{ds \left(\frac{d}{d\sigma}\right)^{-1} f(\sigma)}{[(s+h_1^2)(s+h_2^2)(s+h_3^2)]^{\frac{1}{2}}}, \tag{15}$$

where

$$\sigma = \frac{a^2}{s+h_1^2} + \frac{b^2}{s+h_2^2} + \frac{c^2}{s+h_3^2}. \tag{16}$$

Now the attraction of an ellipsoid of variable density, on an external point, the law of force being that of nature, is expressed by the integral,

$$\iiint \frac{dx\,dy\,dz\,(a-x) f\left(\frac{x^2}{h_1^2} + \frac{y^2}{h_2^2} + \frac{z^2}{h_3^2}\right)}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{\frac{3}{2}}} = -\frac{d\mathrm{v}}{da}$$

$$= h_1 h_2 h_3 \pi \int_0^\infty \frac{ds \frac{d}{da}\left(\frac{d}{d\sigma}\right)^{-1} f(\sigma)}{[(s+h_1^2)(s+h_2^2)(s+h_3)^2]^{\frac{1}{2}}}.$$

But

$$\frac{d}{da} = \frac{d\sigma}{da}\frac{d}{d\sigma} = \frac{2a}{s + h_1^2}\frac{d}{d\sigma},$$

hence,

$$-\frac{dv}{da} = 2\pi h_1 h_2 h_3 \int_0^\infty \frac{ads f(\sigma)}{(s+h_1^2)\,[(s+h_1^2)(s+h_2^2)(s+h_3^2)]^{\frac{1}{2}}}. \qquad (17)$$

To the limits $\sigma = 0$, $\sigma = 1$, correspond the limits $s = \infty$, $s = \lambda$, $\lambda$ being given by the equation

$$\frac{a^2}{\lambda + h_1^2} + \frac{b^2}{\lambda + h_2^2} + \frac{c^2}{\lambda + h_3^2} = 1. \qquad (18)$$

Now the integration in the second member of (17) extends to all positive values of $s$ which make $\sigma$ fall between 0 and 1, i. e. to all positive values of $s$ which lie between $\lambda$ and $\infty$. Let $\frac{a^2}{h_1^2} + \frac{b^2}{h_2^2} + \frac{c^2}{h_3^2}$ be greater than 1, which implies that the attracted point is external; $\lambda$ is then positive, and the limits of $s$ are accordingly $\lambda$ and $\infty$; whence

$$-\frac{dv}{da} = 2a\pi h_1 h_2 h_3 \int_\lambda^\infty \frac{ds f(\sigma)}{(s+h_1^2)\,[(s+h_1^2)\,(s+h_2^2)\,(s+h_3^2)]^{\frac{1}{2}}}. \qquad (19)$$

Again, let $\frac{a^2}{h_1^2} + \frac{b^2}{h_2^2} + \frac{c^2}{h_3^2}$ be equal to or less than 1; $\lambda$ is then 0 or negative, and the limits of $s$ are 0 and $\infty$; whence

$$-\frac{dv}{da} = 2a\pi h_1 h_2 h_3 \int_0^\infty \frac{ds f(\sigma)}{(s+h_1^2)\,[(s+h_1^2)\,(s+h_2^2)\,(s+h_3^2)]^{\frac{1}{2}}}. \qquad (20)$$

This formula determines the attraction in the direction of the axis $x$ when the attracted point is on the surface or is internal.

To deduce from (19) the common expression for the case of external attraction, we must assume

$$\frac{h_1^2 + \lambda}{h_1^2 + s} = u^2;$$

and transforming, we get

$$-\frac{dv}{da} = \frac{4h_1 h_2 h_3 \pi a}{(\lambda + h_1^2)^{\frac{3}{2}}} \int_0^1 \frac{u^2 du f(\sigma)}{\sqrt{(\lambda + h_1^2 + (h_2^2 - h_1^2)u^2)(\lambda + h_1^2 + (h_3^2 - h_1^2)u^2)}}, \qquad (21)$$

in which

$$\sigma = \frac{a^2u^2}{h_1^2+\lambda} + \frac{b^2u^2}{h_1^2+\lambda+(h_2^2-h_1^2)u^2} + \frac{c^2u^2}{h_1^2+\lambda+(h_3^2-h_1^2)u^2}.$$

In the other case we may assume $\frac{h_1^2}{h_1^2+s} = u^2$.

It remains to notice a remarkable class of integrals, the values of which can be assigned in finite algebraic terms. Mr. Cayley was the first to point out their existence, to the discovery of which he was led by performing the operation

$$\frac{d^2}{da_1^2} + \frac{d^2}{da_2^2} \cdot \cdot + \frac{d^2}{da_n^2}$$

on the function which expresses the value of the integral

$$\iint \cdot \cdot \frac{dx_1 dx_2 \,.\,.\, dx_n}{[(a_1-x_1)^2+(a_2-x_2)^2 \,.\,.+(a_n-x_n)^2]^{\frac{n}{2}}},$$

and observing that the single integration which it involves became possible. We may remark, that to this class belong the integrals which express the attractions of homogeneous ellipsoids, when the law of force is any even inverse power of the distance, except the law of gravitation. Shall we say that this is an exception to the observed rule, that the constitution of the material universe, and the possibilities of analysis, are arranged in a certain mutual harmony, or shall we regard it as a fact subservient to some higher principle?

We shall only consider the case of an ellipsoid's attraction when the force varies as the inverse fourth power of the distance. None of the results have, we believe, been given before, and as they are of a very interesting character, we shall dwell upon them at some length; i. e. we shall consider the attractions of ellipsoids of uniform and of variable densities, on both external and internal points.

Now the expression for the attraction parallel to the axis $x$, in this case, is

$$\iiint \frac{dxdydz\,(a-x)f\left(\frac{x^2}{h_1^2}+\frac{y^2}{h_2^2}+\frac{z^2}{h_3^2}\right)}{[(a-x)^2+(b-y)^2+(c-z)^2]^{\frac{5}{2}}} = -\frac{1}{3}\frac{d\mathrm{v}}{da}, \tag{22}$$

where

$$\mathrm{v} = \frac{dxdydz\,f\left(\frac{x^2}{h_1^2}+\frac{y^2}{h_2^2}+\frac{z^2}{h_3^2}\right)}{[(a-x)^2+(b-y)^2+(c-z)^2]^{\frac{3}{2}}}$$

Assuming in (12) $n = 3$, $i = \frac{3}{2}$, we have

$$\mathrm{v} = -\, 2h_1 h_2 h_3 \pi \int_0^\infty \frac{ds f(\sigma)}{s[(s+h_1^2)\,(s+h_2^2)\,(s+h_3^2)]^{\frac{1}{2}}}. \tag{23}$$

In this expression $a$ is only found in the function $\sigma$, *vide* (16); hence

$$\frac{d\mathrm{v}}{da} = \frac{d\sigma}{da}\frac{d\mathrm{v}}{d\sigma} = \frac{2a}{s+h_1^2}\frac{d\mathrm{v}}{d\sigma}$$

$$\therefore\; -\tfrac{1}{3}\frac{d\mathrm{v}}{da} = -\,\frac{4h_1h_2h_3\pi a}{3}\int_0^\infty \frac{ds\,\frac{d}{d\sigma}f(\sigma)}{s(s+h_1^2)[(s+h_1^2)\,(s+h_2^2)\,(s+h_3^2)]^{\frac{1}{2}}}. \tag{24}$$

Suppose, first, the attracted point external, and the density uniform and equal to unity; then, as before, determining $\lambda$ by the equation

$$\frac{a^2}{\lambda+h_1^2} + \frac{b^2}{\lambda+h_2^2} + \frac{c^2}{\lambda+h_3^2} = 1, \tag{25}$$

we see, that in integrating relatively to $s$, we must, when $s$ is less than $\lambda$, regard $f(\sigma)$ as 0, and therefore $\frac{d}{d\sigma}f(\sigma)$ as 0 also; that when $s$ is greater than $\lambda$, $f(\sigma)=1$, and $\frac{d}{d\sigma}f(\sigma)$ is again equal to 0. It appears, therefore, that $\frac{d}{d\sigma}f(\sigma)$ vanishes before and after the break in the discontinuous function $f(\sigma)$. To find its value at the break we must proceed thus. Since $\sigma$ is a function of $s$, we have

$$\frac{d}{d\sigma}f(\sigma) = \frac{ds}{d\sigma}\frac{d}{ds}f(\sigma)$$

$$\therefore\; ds\,\frac{d}{d\sigma}f(\sigma) = \frac{ds}{d\sigma}\frac{d}{ds}f(\sigma)\,ds.$$

Now $\frac{d}{ds}f(\sigma)ds$ is the differential of $f(\sigma)$ relative to $s$, i. e. the difference of the values of $f(\sigma)$ preceding and succeeding the break, and this difference is unity. With the value of $f(\sigma)$ at the break we have nothing to do, as it does not extend to any differential element of $s$. Hence, at the break,

$$ds\frac{d}{d\sigma}f(\sigma) = \frac{ds}{d\sigma}.$$

Now

$$\frac{ds}{d\sigma} = \frac{-1}{\frac{a^2}{(s+h_1^2)^2} + \frac{b^2}{(s+h_2^2)^2} + \frac{c^2}{(s+h_3^2)^2}}.$$

Substituting this in the place of $ds\frac{d}{d\sigma}f(\sigma)$ in (24), changing $s$ into $\lambda$, and rejecting the integral sign, because all the other elements vanish, we have

$$-\tfrac{1}{3}\frac{d\text{v}}{da} = \frac{\frac{3}{4}h_1h_2h_3\pi a}{\lambda(\lambda+h_1^2)[(\lambda+h_1^2)(\lambda+h_2^2)(\lambda+h_3^2)]^{\frac{1}{2}}\left[\frac{a^2}{(\lambda+h_1^2)^2}+\frac{b^2}{(\lambda+h_2^2)^2}+\frac{c^2}{(\lambda+h_3^2)^2}\right]}, \quad (26)$$

which is a finite expression for the attraction of a homogeneous ellipsoid on an external point $abc$ in the direction of $a$, the law of force being $\frac{1}{d^4}$.

If $h_1 = h_2 = h_3 = r$, and if we make $a^2+b^2+c^2 = d^2$, we find $\lambda = d^2 - r^2$, and substituting and reducing

$$-\tfrac{1}{3}\frac{d\text{v}}{da} = \frac{4\pi a r^3}{3d^3(d^2-r^2)}, \quad (27)$$

and this is the result to which we should be led by direct integration, the ellipsoid becoming a sphere.

If $a, b, c$, or any of them, be very great, we find $\lambda = d^2$, and substituting and reducing

$$-\tfrac{1}{3}\frac{d\text{v}}{da} = \frac{4\pi h_1h_2h_3}{3}\frac{a}{d^5} = \frac{\text{A}a}{d^5}, \quad (28)$$

where A is the solidity of the ellipsoid. This shews that at a very great distance the attraction is the same as if the mass of the ellipsoid were collected at its centre.

If $abc$ be a point in the surface of the ellipsoid, $\lambda = 0$, and the expression for the attraction is infinite.

Let us, in the next place, suppose the density variable, the attracted point being still external.

Reasoning as in the last case, we perceive that when $s = \lambda$, the numerator $ds\frac{d}{d\sigma}f(\sigma)$ in (24) assumes a finite value $\frac{ds}{d\sigma}f(1)$. This will give rise to a separate finite term. From $s = \lambda$ to $s = \infty$ the quantity under the sign of integration is continuous. Putting, then, $\frac{d}{d\sigma}f(\sigma) = f'(\sigma)$, we get

$$-\frac{1}{3}\frac{d\text{v}}{da}=\frac{\frac{4}{3}\pi h_1h_2h_3af(1)}{\lambda(\lambda+h_1^2)[(\lambda+h_1^2)(\lambda+h_2^2)(\lambda+h_3^2)]^{\frac{1}{2}}\left[\frac{a^2}{(\lambda+h_1^2)^2}+\frac{b^2}{(\lambda+h_2^2)^2}+\frac{c^2}{(\lambda+h_3^2)^2}\right]}$$

$$-\frac{4}{3}\pi h_1h_2h_3a\int_\lambda^\infty\frac{dsf'(\sigma)}{s(s+h_1^2)[(s+h_1^2)(s+h_2^2)(s+h_3^2)]^{\frac{1}{2}}}. \qquad (29)$$

This expression consists of two parts; the first, which is finite and algebraic, snows what the attraction would be if the density throughout were uniform, and equal to the density at the surface; the second expresses the effect due to the excess of density above, or defect below this uniform state.

When the particle is internal the finite algebraic term must be rejected, and the lower limit of integration in the remaining term must be replaced by 0.

## POSTSCRIPT.

The value of the multiple definite integral,

$$\text{v}=\iint\ldots\frac{dx_1dx_2\ldots f\left(\frac{x_1^2}{h_1^2}+\frac{x_2^2}{h_2^2}\ldots+\frac{x_n^2}{h_n^2}\right)}{[(a_1-x_1)^2+(a_2-x_2)^2\ldots+(a_n-x_n)^2\pm u^2]^i}$$

subject to the same limiting conditions as before, will be expressed in the same form (12), with this difference only, that

$$\sigma=\frac{a_1^2}{s+h_1^2}+\frac{a_2^2}{s+h_2^2}\ldots+\frac{a_n^2}{s+h_n^2}\pm\frac{u^2}{s}.$$

Apparently this is the most general theorem of the kind; for the expression $\Sigma(a+bx+cx^2)$ may be reduced to the form exhibited in the denominator under the index $i$.

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